Challenges in the calibration of real world models within Economic Scenarios Generators

A summary on theory and best practices for interest rates

Hervé Andrès
Pierre-Edouard Arrouy
Paul Bonnefoy
Alexandre Boumezoued
Sophian Mehalla
Julien Vedani

This paper discusses the interest rates (IR) modelling in Real World (RW) environment. It introduces and compares three approaches: the so-called forward-looking, backward-looking, and hybrid techniques. The forward-looking approach leans on the classical Risk Neutral (RN) modelling framework, whereas the backward-looking approach is designed to replicate the stylized facts observed in historical time series. Finally, the hybrid approach is meant to combine the advantages of both approaches.

The RW denomination can cover a wide range of models and calibration processes that can all be very useful depending on the modeller goal. In all generality, a RW measure is supposed to refer to 'price distributions in which market risk preferences are embedded' (Liu et al., 2007). In comparison, the RN measure is based on pure market expectations. These two frameworks are widely used in the insurance industry for financial communication and decision-making. As illustrated by Milliman CHESS,[1] the landscape of economic scenarios generators has grown in terms of sophistication and related regulatory compliance.

RW economic scenarios can be used by insurers for several applications. First, they can be used to perform ALM studies, for instance, in order to find an optimal allocation of their portfolio. Another application is the computation of the Solvency Capital Requirement within an Internal Model by generating scenarios on a one-year horizon and verifying that the insurer can meet its obligation over the simulated year with a 99.5% probability. Insurers also need RW scenarios through a business plan horizon to implement their ORSA process. Finally, RW simulations can be employed to price assets or liabilities that include a risk premium, with a view to perform portfolio transactions (purchase or sale).

Unlike RN modelling, to which special attention was paid in recent years, particularly in terms of calibration and market impact studies (see Arrouy et al., 2017; Vedani et al., 2017; Borel-Mathurin and Vedani, 2019; Arrouy et al., 2019; Andres et al., 2020; and Borel-Mathurin et al., 2020), researches on RW modelling, calibration, and validation remain limited. This paper aims to provide a comprehensive description of different methods dedicated to RW modelling and calibration. More specifically, we present three different approaches of RW modelling and calibration, namely the so-called forward-looking, backward-looking, and hybrid approaches. This paper also opens the door for future studies related to the validation of RW scenarios.

[1] https://fr.milliman.com/fr-fr/products/milliman-chess.

Forward-looking RW modelling – market risk preferences while keeping an equivalence with risk-neutrality

A RW model can be recovered by performing a change of probability measure from an original RN model thanks to Girsanov theorem (Girsanov, 1960). In practice, the idea is to include a (potentially time-dependent) risk premium in the model.

Hull et al., 2014, introduced a quite simple example of this change of measure. Consider a single-factor Markov model for the short rate r, under the RN probability Q:

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t^Q.$$

The drift μ and volatility σ are functions of the time and the short rate. W_t^Q is a Brownian motion under Q. Setting $\mu(r,t) = a(b-r)$ and σ constant yields the well-known Vasicek model. Various alternative choices for the drift and volatility coefficients can be made (see in particular Brigo and Mercurio, 2007, for a thorough survey). Applying Girsanov theorem, one simply gets the associated RW dynamics under the RW probability $\mathcal{P}$:

$$dr_t = \big(\mu(r_t, t) + \lambda(r_t, t)\big)dt + \sigma(r_t, t)dW_t^{\mathcal{P}}.$$

The risk premium λ, reflecting the market price for interest rates risk, can be a function of the time and the short rate and can be calibrated on historical data or future expectations/targets. Note that, in general, forward-looking models therefore still rely on historical data for calculating the risk premiums, while the underlying RN dynamics may also involve the use of historical series through, for example, inter-forward rates correlations calibration within market models, see Arrouy et al., 2017.

A structural property of forward-looking approach is that the volatility process remains the same in RN and RW environments. That might be seen as a strong assumption since the market implied and historical volatilities are different in practice. This can be attributed in large part to the fact that market expectancies are forward-looking estimates not compatible with the historical backward-looking estimates based on time-series and historical statistics. This link between historical and implied volatilities has been studied in a relatively limited number of references, see Christensen and Hansen, 2002.

The forward-looking approach has the advantage of preserving the consistency between RN and RW modelling. It may be particularly convenient for some applications, such as when coherent forecasts are required for both realized (level of equity and interest rates) and implied RN quantities (implied volatility, implied default probabilities). Moreover, it allows to leverage the extensive literature on RN calibration. Nevertheless, forward-looking approaches suffer from major drawbacks as they fail to replicate main stylized facts, such as the asymmetry or the non-stationarity of the historical time series. This can be an issue for some applications such as portfolio allocation or economic capital calculation, from both a methodological and regulatory standpoint.

Backward-looking RW approach

As an alternative to forward-looking methods, the RW measure can be induced by a backward-looking model calibration relying on time series. Such an approach appears to be relevant to capture the key historical features of the risk driver using a proper statistical approach, in order to derive future scenarios that are coherent with the historical observations. This is a strong Bayesian assumption that one possesses a useful *a priori* on what is going to happen in future years. Indeed the underlying processes (yields, returns,…) are barely stationary.[2]

Nevertheless, this approach is often used to capture realism in projections. For example, as far as IR models are concerned, a mean-reversion model can be useful for long-term projections. Another possibility can be the use of expertise of asset management specialists to introduce *a priori* targeted values for the simulated quantities (e.g., middle or long-term mean targets for IR and for indices). Note that by nature, backward-looking models don't necessarily have risk neutrality even as a limiting case of the parameters, for example, they aren't necessarily 'term structure models' in the sense used in risk-neutral formulations.

This practical backward-looking approach is currently used for multiple applications, including economic capital models and asset allocation.

[2] As presented below, some alternative (recent and still in progress) backward-looking approaches using fractional Brownian motions allow to simulate non-stationary RW processes.

EXAMPLE OF BACKWARD-LOOKING IR MODELS

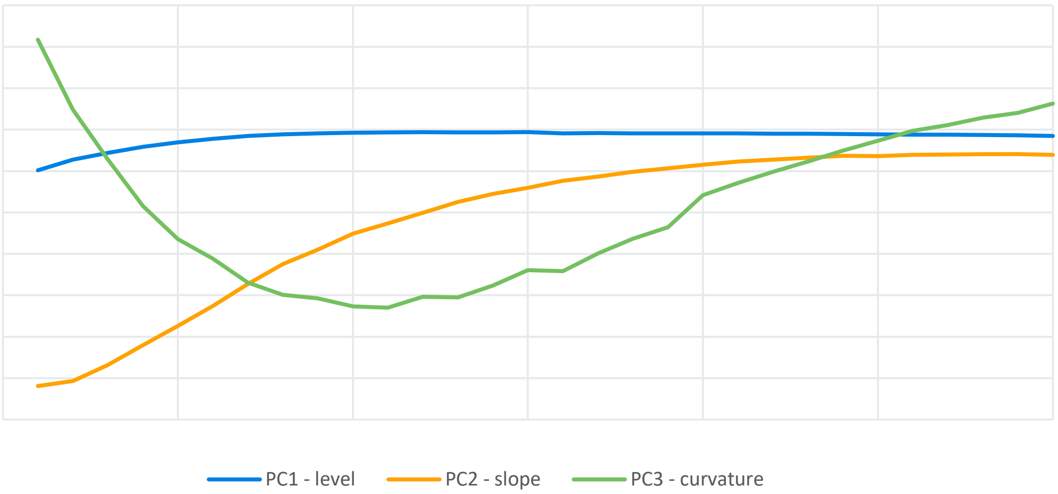
Key criterion when selecting an RW model for any projection is to compare various possibilities depending on the projections realism and the goal of the user.

In all generality, a yield curve (e.g., for maturities 1 to 30) can be largely explained as a linear combination of three principal components, the so-called level, slope, and curvature. These are easily captured through a Principal Component Analysis (PCA). In order to model quantities that are historically stationary, the PCA is generally applied on a transformation of the rates curve. For example, one can work on the log-ratios or the differences. In the former case, a shift can also be applied in order to be able to model negative rates. If one considers a model that describes the rates curve through its variations (log-ratios or differences), then the PCA writes:

$$\Delta r_m(t) = \alpha c_m^1 + \beta c_m^2 + \gamma c_m^3,$$

where $\Delta r_m(t)$ is the variation of the interest rate of maturity m, (α, β, γ) are the three parameters defining the model and where $(c_m^1)_{1 \le m \le 30}$ is the first "level" component, $(c_m^2)_{1 \le m \le 30}$ is the second 'slope' component and $(c_m^3)_{1 \le m \le 30}$ is the third 'curvature' component.

FIGURE 1: EXAMPLE OF HISTORICAL IR PRINCIPAL COMPONENTS (3 FIRST PRINCIPAL COMPONENTS OF A PCA)

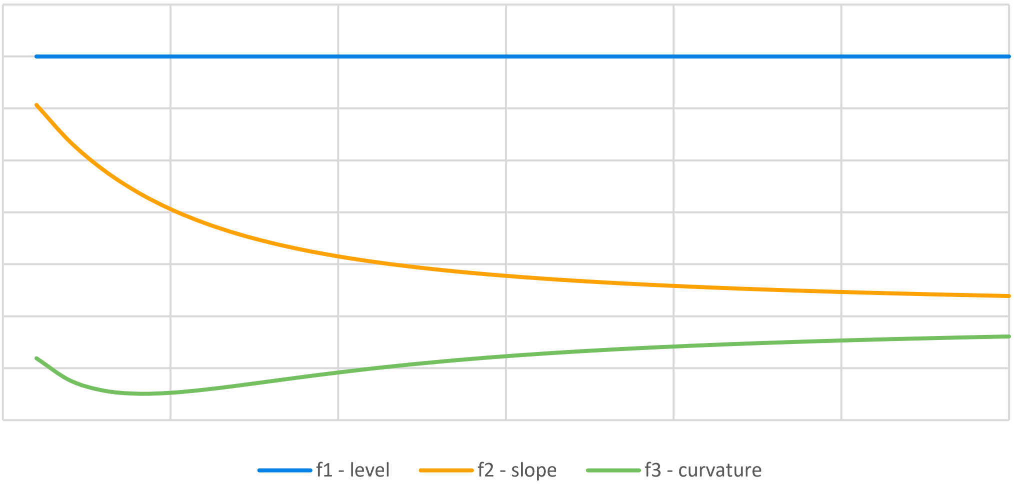


This simplification leads to various projection models for the interest rates $r_m(t)$. They differ in the way of modelling the series $(\alpha_t, \beta_t, \gamma_t)_t$ (e.g., prescribing a Vasicek dynamics for each coefficient) so the complexity is limited to a fixed number of random factors (three factors in the previous example). An alternative simplification scheme is the Nelson-Siegel (NS) model used in particular in Diebold and Li, 2005. Under the NS model, the curve is parametric and relies on four parameters $(\beta_0, \beta_1, \beta_3, \lambda)$:

$$\Delta r_m = \beta_0 + \beta_1 \frac{1 - e^{-\frac{m}{\lambda}}}{\frac{m}{\lambda}} + \beta_2 \left(\frac{1 - e^{-\frac{m}{\lambda}}}{\frac{m}{\lambda}} - e^{-\frac{m}{\lambda}} \right).$$

In practice, once λ is calibrated, this scheme writes as a version of PCA modelling with components $f_m^1 = 1$, $f_m^2 = \frac{1-e^{-\frac{m}{\lambda}}}{\frac{m}{\lambda}}$ and $f_m^3 = \frac{1-e^{-\frac{m}{\lambda}}}{\frac{m}{\lambda}} - e^{-\frac{m}{\lambda}}$.

FIGURE 2: EXAMPLE OF NELSON-SIEGEL FUNCTIONALS ($\lambda = 2.337$)



As one can see, the Nelson-Siegel and PCA components have similar shapes, with some differences however for the second principal component. Similarly to the PCA scheme, it is possible to use this parametric framework and a model to project the $(\beta_0, \beta_1, \beta_2)$ as a time series based on a given stochastic diffusion (e.g., Vasicek model) in order to recover RW forecasts for $r_m(t)$. It is possible to consider an additional second curvature component which defines the Nelson-Siegel-Svensson (see Svensson, 2005) model. It is in use, for example, by central banks such as the Federal Reserve (see Gurkaynak et al., 2007) or the European Central Bank (see Coroneo et al., 2011).

Let us make two remarks before moving to the calibration of backward-looking models.

First, the two provided examples can lead one to think that the backward-looking approach reduces to PCA models but the backward-looking approach is in fact more general. It includes any approach which is driven by historical data and is able to reproduce its main characteristics. In this section, we took PCA models as example since they have a strong ability to catch the general aspect of historical data. In fact, PCA models are not only used within backward-looking approaches but also within forward-looking approaches. In particular, PCA can be used in RN market-consistent IR models, to assess inter-forward rates correlations (e.g., in Libor Market Models with or without displaced diffusion and stochastic volatility).

Second, the use of some backward-looking RW models, in particular those without mean-reversion or prescribed volatility targets can lead to uncontrolled simulations of yield curves, especially for middle- to long-term horizons. This aspect is not essential[3] for some applications but simulations should be checked to verify that yield curves do not diverge 'too far' away from term structure explanation.

CALIBRATING THE MODEL

The model calibration may seem straightforward but can raise some questions. The backward-looking calibration requires a set of historical data and to optimize the parameters to best fit these data. The optimization program is generally either a likelihood maximization or a least squares program (see James & Webber, 2000).

The calibrated parameters may be highly dependent on the considered time-period of the data but also on the time-step between data points.

- Regarding the **considered time-period**, it is well known that financial data are barely stationary due in particular to specific stressed periods (bubbles, crises, national or international economic uncertainty,…) leading to a non-stationary volatility of data through time. To integrate this variation in stochastic behaviour of IR, leading to structural breaks in time series, it is possible to consider, and test, regime-switching models (see Gray, 1996; Ang and Bekaert, 2002). These models have shown good results to project short-term IR but the number of parameters is increasing, which often leads to a lack of stability.

[3] The first AAA RW-generator used in the U.S. also didn't have a mean-reversion property in its diffusion model.

- Overall, best practice would be to achieve an optimal tradeoff between using the largest historical dataset available and relying on relatively recent conditions that will better reflect future forecasts. In addition, a large set introduces stability in the calibrated parameters, which is valuable when the calibration set is regularly updated with most recent data.
- Considering the **time step**, the lack of stationarity can lead to great differences in estimation and simulation. The issue is linked with an autocorrelation of historical financial processes, a property that cannot be captured using Brownian motions to project future evolutions of interest rates. The autocovariance of a standard Brownian Motion $(B_u)_{u\geq 0}$ is:

$$\mathbb{E}[B_s B_t] = \min(s,t) \quad \forall s,t \in \mathbb{R}^+,$$

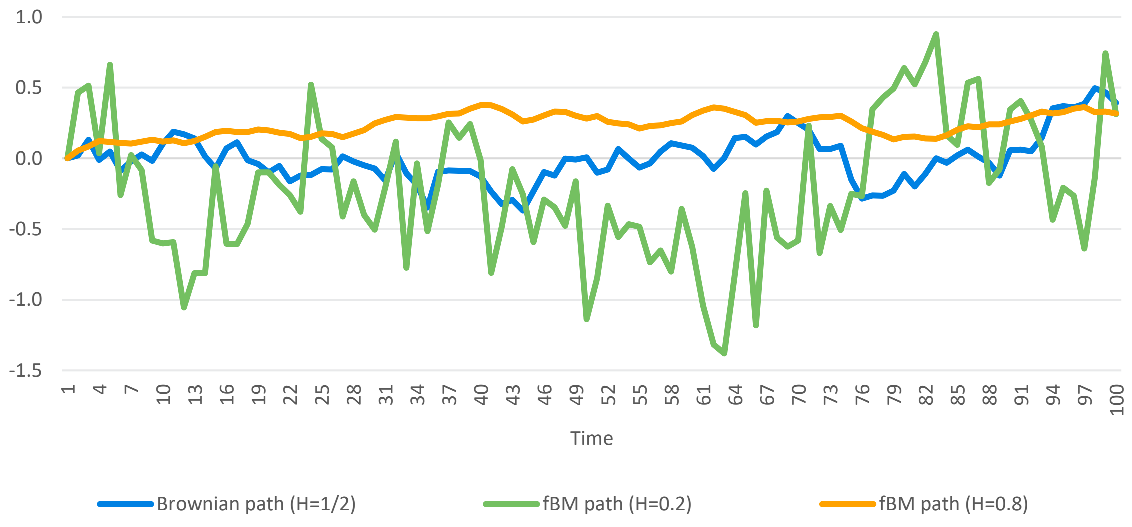
which is not consistent with real market data. This problem is partly solved using fractional Brownian motions (fBM, see Sottinen, 2001; Oksendal, 2003). These processes can better capture the historical autocovariance of IR increments through a specific parameter, the Hurst index $H \in (0,1)$ that account for more general autocorrelation functions. In particular, choosing $H > \frac{1}{2}$ allows to model processes with long memory. Denoting a fBM of Hurst index H by $(B_u^H)_{u\geq 0}$, we have:

$$\mathbb{E}[B_s^H, B_t^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Observe that the standard Brownian motion can be recovered with $H = \frac{1}{2}$. The H index has a great impact on the volatility process as shown in the figure below.

In their seminal paper, Gatheral et al., 2018, showed that the log-volatility of the main stock indexes essentially behaves as a fBM with Hurst index of order 0.1. Since then, models based on fBM have become very popular and represent an active field of research in financial mathematics. Besides, fractional models don't generate trajectories presenting volatility clustering, which is largely acknowledged as a stylized fact in historical data. Instead, GARCH or regime-switching models allow describing this phenomenon. Once again, there is no simple answer and any choice should be made under conservative motivations.

FIGURE 3: FBM PATHS ILLUSTRATION



These backward-looking approaches show two major drawbacks. The first one is to assume distribution stationarity for future scenarios, framed by what happened in the past. The second is a limited ability of these models in their standard form to capture distributional targets, for example, from asset management experts, when the model calibration is fully driven by historical data. As a particular example, portfolio allocation is an area which benefits from the value of expert judgement in setting coherent assumptions, for example, for return on risk across asset classes, in view of avoiding outliers that may skew results.

Towards hybrid method

Hybrid approaches expand the role of history in calibrations, beyond the forward-looking case, but make a point of maintaining the connection to the risk-neutral limit, which remains a limiting case for a certain parameterization. In other words, the hybrid approach aims to combine the advantages of both the forward-looking and the backward-looking techniques:

1. By retaining the core structure of models that have been successfully used in RN modelling, due to their tractability, and as such preserving a coherence between the RN and the RW universe (forward-looking property)
2. By calibrating the parameters of the dynamics that drive the distributional properties of the model on past data (backward-looking character) to make the scenarios consistent with the observed historical features (volatility, dispersion, correlations…)

Let us consider as an example the two factors Gaussian model (G2++ for short; see Brigo and Mercurio, 2007, for more details) used to simulate the short rate r_t:

$$r_t = x_t + y_t + \varphi(t), r_0 = r_0^*,$$

where the processes $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ satisfy:

$$\mathrm{d}x_t = -ax_t\mathrm{d}t + \sigma dW_t^1, x_0 = 0,$$

$$\mathrm{d}y_t = -by_t\mathrm{d}t + \eta dW_t^2, y_0 = 0,$$

with $\varphi(t)$, the deterministic function used to fit the initial yield curve, $(W_t^1)_{t\geq 0}$ and $(W_t^2)_{t\geq 0}$ two correlated Brownian motions under the RN probability Q, such that $dW_t^1 \cdot dW_t^2 = \rho dt$. When changing measure to RW measure this diffusion leads to:

$$r_t = x_t + y_t + \varphi(t), r_0 = r_0^*,$$

where the processes $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ satisfy:

$$\mathrm{d}x_t = -ax_t\mathrm{d}t + \sigma\lambda^1(t)dt + \sigma dB_t^1, x_0 = 0,$$

$$\mathrm{d}y_t = -by_t\mathrm{d}t + \eta\lambda^2(t)dt + \eta dB_t^2, y_0 = 0,$$

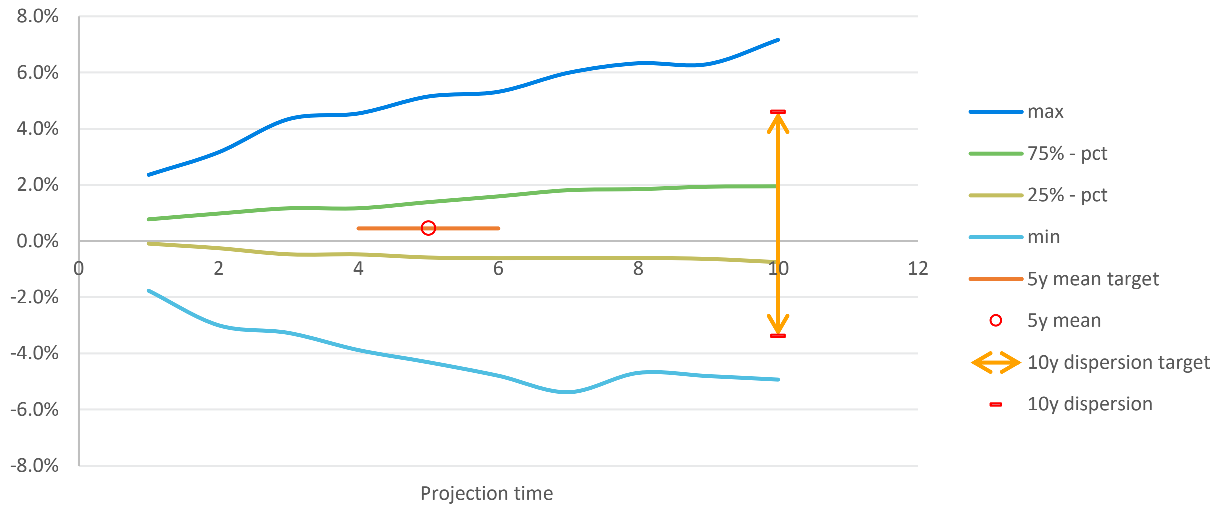
with $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ two correlated Brownian motions under the RW probability $\mathcal{P}$ and where $\lambda(t) = \left(\lambda^1(t), \lambda^2(t)\right)$ is the risk premium assumed deterministic. The correlation stays the same under $\mathcal{P}$: $dB_t^1 \cdot dB_t^2 = \rho dt$.

Under $\mathcal{P}$ or Q it is possible to derive closed-form formulas for the volatility, dispersion, and correlation of interest rates using Ito's isometry. It should be noted that these formulas only depend on the so-called distributional parameters $(a, b, \sigma, \eta, \rho)$ and not on the risk premium or the function φ (it is an advantage of using a Gaussian model). The volatility, dispersion and correlation within the G2++ model can then be compared to the same quantities estimated on the considered historical dataset. This enables a calibration of parameters $(a, b, \sigma, \eta, \rho)$. Then, using these parameters it is easy to calibrate a parametrization of the functions $\varphi(t)$ and $\lambda(t)$. This leads to a three-step approach:

- Step 1: Calibration of parameters $(a, b, \sigma, \eta, \rho)$ based on historical targets (volatilities, distribution dispersion, and historical correlations).
- Step 2: Assessment of the function $\varphi(t)$ to replicate the initial yield curve (e.g., EIOPA RFR disclosure). For this purpose, we work again under the RN probability Q in order to be able to use the ordinary pricing formulas of the zero-coupon bonds.
- Step 3: Once all other parameters are estimated, use of economic expectations on long-term interest rates levels to estimate $\lambda(t) = \left(\lambda^1(t), \lambda^2(t)\right)$.

The figure below depicts the shape of simulated rates (1,000 random paths of Zero-Coupon rates of five-year maturity) using this hybrid method. In particular, the target mean and dispersion (orange lines and arrow) are compared with simulated quantities (red items).

FIGURE 4: HYBRID APPROACH ILLUSTRATION: THE STATISTICAL DISTRIBUTION OF 10 YEARS RW IR SIMULATIONS WITH 5-YEAR MATURITY ARE REPRESENTED.



Before concluding, we summarize in the table below the pros and cons of the three presented approaches:

	Pros	Cons
Forward-looking	▪ Consistent with RN models ▪ Allows reuse of the calibrated parameters in an RN framework	▪ Fails to replicate stylized facts observed on historical time series
Backward-looking	▪ Allows replication of historical stylized facts	▪ Inconsistent with RN models
Hybrid	▪ Consistent with RN models ▪ Allows replication of historical stylized facts	▪ Possibly, less realistic shapes than those from the backward-looking approach

Conclusion

In practice, no specific RW framework is regulatory compulsory for insurance undertakings be it for short term (internal model one-year projections) or long term (ORSA projections, business plan horizon). A forward-looking, backward-looking, or hybrid model and calibration may be used as far as it is well justified, produces realistic evolutions and is consistent with the modelling objective. Overall, a legitimate model must be well understood by actuaries. Its parameters' stability through time should be checked, and all movements should lead to impact studies and level sets that may be plotted, used, and justified.

We recognize that the best 'one-size-fits-all' ESG solution is the one that provides the choice of different methods that work better for different applications / economic environments. For more insights, visit https://www.milliman.com/en/products/milliman-chess.

Milliman

Milliman is among the world's largest providers of actuarial and related products and services. The firm has consulting practices in life insurance and financial services, property & casualty insurance, healthcare, and employee benefits. Founded in 1947, Milliman is an independent firm with offices in major cities around the globe.

milliman.com

CONTACT

If you have any questions or comments on this paper or any other aspect of Economic Scenarios Generation, please contact your usual Milliman consultant or the email address below:
chess@milliman.com

REFERENCES

Andres, Arrouy, Bonnefoy, Boumezoued, Mehalla (2020). Fast calibration of the LIBOR Market Model with Stochastic Volatility based on analytical gradient. Submitted. https://arxiv.org/pdf/2006.13521.

Ang, A. & Bekaert, G. (2002). Regime switches in interest rates. Journal of Business & Economic Statistics, 20(2),. 163-182.

Arrouy, Boumezoued, Lapeyre, Mehalla (2019). Fast calibration of the LIBOR Market Model with Stochastic Volatility based on a Jacobi process approximation. Finance and Stochastics. https://hal.archives-ouvertes.fr/hal-02468583v1.

Borel-Mathurin, F., El Karoui, N., Loisel, S., & Vedani, J. (2020). Locality in time of the European insurance regulation 'risk-neutral' valuation framework, a pre- and post-Covid analysis and further developments. https://hal.archives-ouvertes.fr/hal-02905181.

Borel-Mathurin, F., & Vedani, J. (2019). Market-consistent valuation: a step towards calculation stability. https://acpr.banque-france.fr/sites/default/files/190827_oppaper_fbm_jv.pdf.

Brigo, D. & Mercurio, F. (2007). Interest rate models-theory and practice: with smile, inflation, and credit. Springer Science & Business Media.

Christensen, B. J., & Hansen, C. S. (2002). New evidence on the implied-realized volatility relation. The European Journal of Finance, 8(2), 187-205.

Coroneo, L., Nyholm, K., & Vidova-Koleva, R. (2011). How arbitrage-free is the Nelson–Siegel model? Journal of Empirical Finance, 18(3), 393-407.

Arrouy, P.E., Bonnefoy, P., Boumezoued, A., & Devineau, L. (2017). Fast calibration of the LIBOR Market Model with Stochastic Volatility and Displaced Diffusion. Journal of Industrial and Management Optimization. https://arxiv.org/pdf/1706.00263.

Diebold, F.X. & Li, C. (2006). Forecasting the term structure of government bond yields. Journal of econometrics, 130(2), 337-364.

Gatheral, J., Jaisson, T., & Rosenbaum, M. (2018). Volatility is rough. Quantitative Finance, 18(6), 933-949.

Girsanov, I.V. (1960). On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. Theory of Probability & Its Applications, 5(3), 285-301.

Gray, S.F. (1996). Modelling the conditional distribution of interest rates as a regime-switching process. Journal of Financial Economics, 42(1), 27-62.

Gurkaynak, R.S., Sack, B., & Wright, J. H. (2007). The U.S. Treasury yield curve: 1961 to the present. Journal of monetary Economics, 54(8), 2291-2304.

Hull, J., Sokol, A., & White, A. (2014). Short-rate joint-measure models. Risk, 59.

James, J. & Webber, N. (2000). Interest rate modelling. Wiley

Liu, X., Shackelton, M.B., Taylor, S.J., & Xu, X. (2007). Closed-form transformations from risk-neutral to real-world distributions. Journal of Banking & Finance, 31(5), 1501-1520.

Oksendal, B. (2003). Fractional Brownian motion in finance. Preprint series. Pure mathematics http://urn. nb. no/URN: NBN: no-8076.

Sottinen, T. (2001). Fractional Brownian motion, random walks and binary market models. Finance and Stochastics, 5(3), 343-355.

Svensson, L.E.O. (1994). Estimating and interpreting forward interest rates: Sweden 1992-1994. National bureau of economic research.

Vedani, J., El Karoui, N., Loisel, S., & Prigent, J. L. (2017). Market inconsistencies of market-consistent European life insurance economic valuations: pitfalls and practical solutions. European Actuarial Journal, 7(1), 1-28. https://hal.archives-ouvertes.fr/hal-01242023/file/LMCPM__v9b_111215.pdf.